Financial Applications of the Mahalanobis Distance

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Motivation

- Many problems in finance involve multivariate random variables $\Rightarrow$ Similarity of realizations?

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- Many problems in finance involve multivariate random variables $\Rightarrow$ Similarity of realizations?
- $\Rightarrow$ Suitable (based on statistical theory) measure for detection: the Mahalanobis distance (MD)
- Examples: outlier detection, portfolio surveillance, asset classification

Introduction

- Application based on MD: Financial Turbulence [Kritzmann & Li, 2010]
- Multivariate unusualness in financial market data

$$FT_t = (\mathbf{r}_t - \boldsymbol{\mu})' \, \Sigma^{-1} (\mathbf{r}_t - \boldsymbol{\mu}) \, . \tag{1}$$

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- $\Rightarrow$ Based on squared MD
- $\mathbf{r}_t$, $\boldsymbol{\mu}$ and Σ may each be determined or estimated in various different ways.

Introduction

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- $\Rightarrow$ Based on squared MD
- $\mathbf{r}_t$, $\boldsymbol{\mu}$ and Σ may each be determined or estimated in various different ways.
- $\Rightarrow$ Depending on inputs, the resulting Mahalanobis distance will have a different economic interpretation.

Aim of the paper:

Explore promising combinations (Input).
Discuss previous and potential uses and usefulness for financial market participants (Output).

Contents

Initial Motivation for MD

- Analyze and classify human skulls into groups, based on different properties [Mahalanobis, 1927]
- Properties for group classification in a financial context:
  Returns on assets in a portfolio
  Portfolio properties for an investment company

Use of MD suggested by Kritzmann & Li (2010)

- MD as indicator for unusualness in financial markets (Financial Turbulence)
- Return of one asset in relation to it's historical mean and standard deviation: $(r_t - \mu)^2/\sigma^2$
  $\Rightarrow$ (Squared) Mahalanobis Distance for one asset

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- Multivariate extension in a portfolio context

$$\text{Eu}_t^2 = \sum_{i=1}^{n} \frac{w_i^2}{\mathbf{w}^2} \frac{(r_{t,i} - \mu_i)^2}{\sigma_i^2} \tag{2}$$

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  $\Rightarrow$ Weighted, squared and normalized Euclidean Distance
- Including information on the direction of moves

$$\mathrm{Ma}_t^2 = \frac{1}{\mathbf{w}^2} \left(\mathbf{r}_t - \boldsymbol{\mu}\right)' \mathbf{w}_D \Sigma^{-1} \mathbf{w}_D \left(\mathbf{r}_t - \boldsymbol{\mu}\right) \tag{3}$$

  $\Rightarrow$ Weighted, squared and normalized Mahalanobis Distance (Portfolio Turbulence)
  $\Rightarrow$ Additionally treats joint deviations of portfolio returns r_i and r_j given by their correlation ρ_{ij}

(Statistical) Properties of the MD

- (Squared) MD has a $\chi^2(n)$-distribution (given $\mathbf{r}_t \sim N_n(\mu, \Sigma)$)

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- (Weighted & squared) MD has a generalized χ^2-distribution with parameters Σ and $\mathbf{w}_D \Sigma^{-1} \mathbf{w}_D$ and Expectation $\mathbf{w}^2 = tr(\mathbf{w}_D^2) = \sum_{i=1}^{n} w_i^2$
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- Normalization leads to an Expaction of 1
- MD is invariant under affine transformation $Y = a + B\dot{X}$ [Meucci, 2009]
- MD captures all statistical information for elliptical distributions (fully described by location parameter μ and scatter matrix Σ)

Contents

Sample-based input paramters

To detect major changes in markets

Distinguish two types of sample based differences:

- Compare a realization to a historical sample or
- Compare two samples

Sample-based input paramters

Compare a realization to a historical sample

- Financial and Portfolio turbulence: relates today's returns to historical means ($\boldsymbol{\mu} = \bar{\mathbf{x}}$) and sample covariances ($\Sigma = S$).
- $\Rightarrow$ Standardized indicator of unusual behavior across markets and portfolios.
- Useful for market/portfolio surveillance [Bodnar, 2009].

Sample-based input paramters

Compare two samples

- Compare two samples of different time periods by relating their means and covariance matrices against each other [McLachlan, 1999]
  ⇒ Hotelling T^2-test for multivariate dependent samples, based on (squared) MD [Rao, 2009].

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Sample-based input paramters

Compare two samples

- Compare two samples of different time periods by relating their means and covariance matrices against each other [McLachlan, 1999]
  $\Rightarrow$ Hotelling T^2-test for multivariate dependent samples, based on (squared) MD [Rao, 2009].
- Use both methods to determine periods of (non-) turbulence.
- Use turbulent market parameters for stress testing portfolios [Chow, 1999].
- Use non-turbulent market parameters (removing outliers) for robust portfolio estimation [Campbell, 1998].

Sample-based input paramters

Example 1: Change in market conditions during the financial crisis

- Portfolio equities (3/8), fixed income (2/8), real estate (2/8) and alternative investments (1/8)
- Historical sample calibration period: 2004-2006

	mean	**std.dev**	**skewness**	**kurtosis**
Equities	0.0006	0.0058	-0.23	1.02
Fixed income	0.0001	0.0035	0.00	1.01
Real estate	0.0010	0.0067	-0.49	1.40
Alt. investments	0.0003	0.0146	0.13	0.43

Table 1: Descriptive statistics - Time frame: 01/2004-12/2006. Indices to proxy for these asset classes: Equities: FTSE ALL WORLD, Fixed income: Barclays Multiverse All, Real estate: FTSE EPRA/NAREIT Global, Alt. investments (proxied using commodities): S&P GSCI Commodities

Sample-based input paramters

Example 1 cont.: Change in market conditions during the financial crisis

- As of 01/2007 observe the MD for an indication of changed market conditions

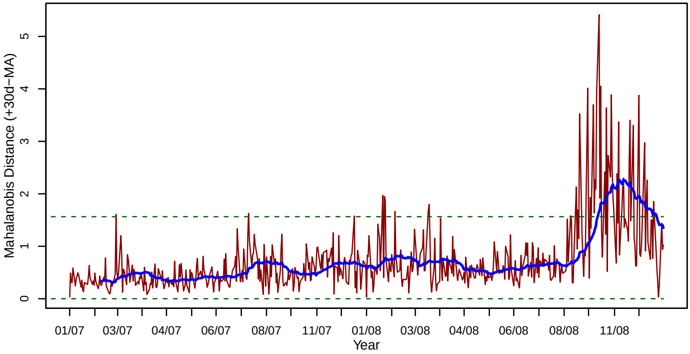

Figure 1: Portfolio Turbulence: MD of realized returns from historical returns (sample period 01/2004-12/2006) and 0.05/0.95 confidence bands (green)

Deviations from Model Prices

Use the MD to find deviations of observed from implied returns of theoretical/empirical models such as the CAPM or the Fama-French 3-factor model

Use. . .

- . . . to asses how much markets are in line with models
- . . . as indication for bubbles in markets
- . . . to assess which competing model best describes the markets

Deviations from Model Prices

Example 2: Asset returns implied by the Black-Litterman Model [Black & Litterman, 1992]

- Use monthly returns and market capitalization of 22 country indices from 1988-2012 [Kaiser et al., 2013]
- Calculate implied returns $E(\mathbf{r}_{t+1}) = \delta \Sigma_t \mathbf{w}$ [Walter, 2011], assuming $\delta = 1$, weights $\mathbf{w}_t$ and covariance matrix Σ_t (120-month rolling window)

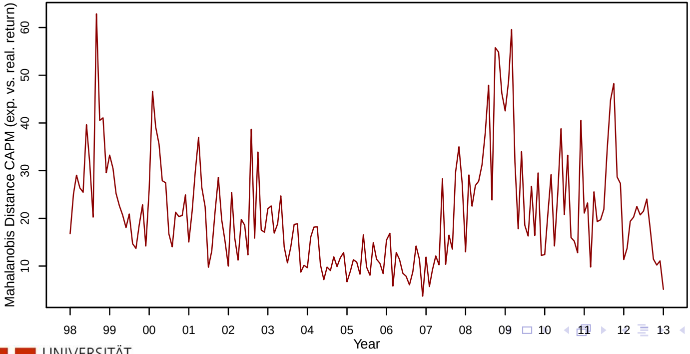

Figure 2: Mahalanobis distances of realized vs. Black/Litterman-implied values

Forecast Evaluation

Use of the MD to evaluate multivariate forecasts

- For a multivariate point forecast $\boldsymbol{\mu}_t$, its MSE-Error matrix Σ determines a MD-based confidence ellipsoid $\Rightarrow$ to evaluate the forecast quality regarding the observation $\mathbf{r}_t$
- The MSE-matrix determines the shape of the confidence ellipsoid (defined by the MD) with center $\boldsymbol{\mu}_t$.

Forecast Evaluation

Two possibles uses of the MD

Possible uses of the MD

1. To calculate forecast confidence regions for a (univariate) forecast path - important for many (path-dependent) financial applications [Jorda & Marcellino, 2010] (Wald Statistic based on MD)
2. Evaluate multivariate point-forecast using confidence ellipsoids [Lütkepohl, 2006] (in contrast to "Bonferroni"-confidence rectangles)

Forecast Evaluation

Example 3: Estimating a VAR-model for exchange rates

- VAR-model based on log-differences of the USD/GBP spot rate $ds_t = \Delta log(s_t)$, and the resp. forward premium $p_t = f_t - s_t$ (01/1980 – 07/2013)
- Estimation yields the following VAR(1,1)-proccess:

$$\begin{pmatrix} ds_t \\ p_t \end{pmatrix} = \begin{pmatrix} -0.0029 \\ -0.0002 \end{pmatrix} + \begin{pmatrix} 0.0671 & -0.8373 \\ 0.0004 & 0.9025 \end{pmatrix} \begin{pmatrix} ds_{t-1} \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \qquad (4)$$

with $\Sigma_u = \begin{pmatrix} 0.000888 & -0.000008 \\ -0.000008 & 0.000002 \end{pmatrix}$.

- MSE-matrices $\Sigma_y(1) = \Sigma_u$ and $\Sigma_y(2) = \Sigma_y(1) + \Phi_1 \Sigma_y(1) \Phi_1' = \begin{pmatrix} 0.00089 & -0.00006 \\ -0.00006 & 0.00064 \end{pmatrix}$ [Lütkepohl, 2006]

Forecast Evaluation

Example 3 cont.: Estimating a VAR-model for exchange rates

Table 2: VAR forecasts vs. realized values

Forecasts	ds_{t+h}	p_{t+h}	$ds_t(h)$	$p_t(h)$
06/2013	-0.0262	-0.0003	-0.0009	-0.0005
07/2013	0.0004	-0.0003	-0.0025	-0.0006

MSE $\Sigma_y(h)$	2013M06		2013M07	
	0.000888	-0.000008	0.00089	-0.00006
	-0.000008	0.000002	-0.00006	0.00064

This table reports forecasts (for 06/2013 and 07/2013) for the log return of the USD/GBP spot rate (ds) and the forward premium (p), using the VAR process stated above. In addition, it shows the realized values for these variables together with the corresponding MSE matrices.

Forecast Evaluation

Example 3 cont.: Estimating a VAR-model for exchange rates

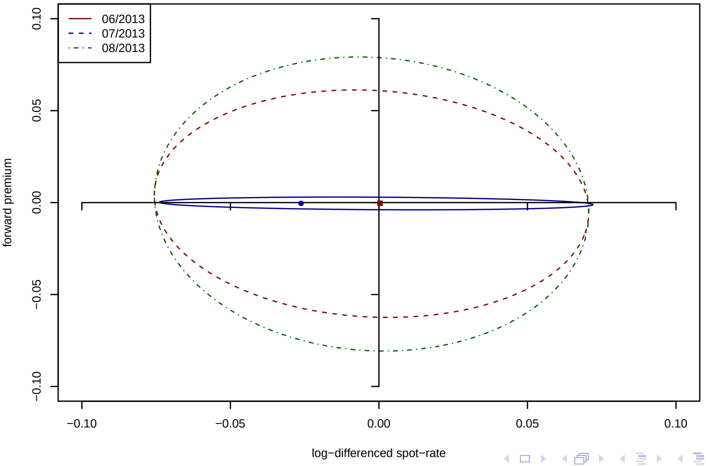

Figure 3: Forecast confidence ellipses based on the Mahalanobis distance

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Conclusion

For multivariate (financial) problems

MD naturally supports answering questions regarding existence and magnitude of deviations

- between observations
- between observations and theoretically (model-) implied values
- between observations and predictions

implying different economic interpretations.

The end

Thank you . . .

. . . very much for your attention!

Literature I